A General Analysis of Corrections to the Standard See-saw Formula in Grand Unified Models

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Abstract

In realistic grand unified models there are typically extra vectorlike matter multiplets at the GUT scale that are needed to explain the family hierarchy. These contain neutrinos that, when integrated out, can modify the usual neutrino see-saw formula. A general analysis is given. It is noted that such modifications can explain why the neutrinos do not exhibit a strong family hierarchy like the other types of fermions.

1 Introduction

It is well-known that in simple grand unified theories (GUTs) the masses of the light neutrinos are given by the see-saw formula

$$M_{\nu\nu} = -m_\nu \mathcal{M}^{-1} m_\nu^T, \tag{1}$$

where m_ν is the Dirac mass matrix connecting left- and right-handed neutrinos and $\mathcal{M}$ is the Majorana mass matrix of the right-handed neutrinos [1]. Typically, the Dirac matrix m_ν is closely related by the unified symmetries to the Dirac mass matrices of the quarks and charged leptons, and like them exhibits a "hierarchical" pattern. This presents something of a puzzle,

since the evidence suggests that $M_{\nu\nu}$ does not have a strongly hierarchical structure. (In particular, the neutrino mixing angles θ_{atm} and θ_{sol} are not small, and the mass ratio $\sqrt{\delta m^2_{sol}}/\sqrt{\delta m^2_{atm}}$ is about 0.2 [2] [3].) One way to reconcile these facts is to posit a hierarchy in $\mathcal{M}$ that is just such as to nearly cancel the hierarchy in m_ν. However, this idea seems contrived and is not easy to implement in a convincing way.

Another possibility is that $M_{\nu\nu}$ receives other contributions besides the usual see-saw term given in Eq. (1). This can happen if the Higgs multiplets that break $B-L$ at the GUT scale and generate $\mathcal{M}$ also contain VEVs that break the weak interaction $SU(2)$. This leads to the so-called type II [4] and type III [5] see-saw contributions. In this paper, we point out a different kind of modification of the see-saw formula that comes from the existence of the superheavy vectorlike fermion multiplets that are quite a typical feature of realistic grand unified models. We give a general analysis of the form of $M_{\nu\nu}$ in grand unified theories, showing under what conditions the different kinds of terms arise.

The standard analysis that leads to Eq. (1) considers the case where there is a left-handed neutrino (ν_i) and a right-handed neutrino (N^c_i) for each family ($i = 1, 2, 3$). We will be interested in unified groups for which ν_i and N^c_i are contained in a single irreducible multiplet $\mathbf{F}_i$. An obvious example is $SO(10)$, where ν_i and N^c_i are contained in the spinor $\mathbf{16}_i$. In fact, we shall carry out the analysis in $SO(10)$, though it will apply more generally. The crucial point that makes it necessary to go beyond the standard analysis is that in realistic unified models there are usually *additional* multiplets of quarks and leptons besides the three familes $\mathbf{F}_i$. A very strong reason to suppose that extra fermions exist is the need to explain the hierarchical pattern of the mass matrices of the quarks and leptons, for the small entries of these matrices are generally assumed to arise from higher-dimensional effective operators obtained by integrating out such extra fermions at the GUT scale [6]. In section 2 we will review the aspects of this idea that are relevant to our subsequent analysis.

Since such extra fermions must get superlarge masses, they necessarily are contained in a real (or "vectorlike") set of multiplets, i.e. in real irreducible multiplets and/or in conjugate pairs of complex multiplets of the unified group. For example, in $SO(10)$ they can be contained in vector and tensor multiplets ($\mathbf{10}$, $\mathbf{45}$, etc.) or in conjugate pairs of spinor and antispinor ($\mathbf{16}+$

$\overline{\mathbf{16}}$, etc.). Let us call this real set of multiplets $\mathbf{R}$. Thus the total quark and lepton content is contained in $\mathbf{F}_{i=1,2,3} + \mathbf{R}$. Typically, one expects that there will be in $\mathbf{R}$ both left-handed neutrinos plus their conjugates (which we shall denote by $\nu'_a + \overline{\nu}'_a$) and right-handed neutrinos plus their conjugates (which we shall denote by $N^{c\prime}_a + \overline{N^c}'_a$). Clearly the general situation for neutrino masses becomes much more complicated than usually assumed. Nevertheless, a general analysis is possible that leads to some simple conclusions. It can be shown under what conditions corrections to the usual see-saw formula arise, and what their form is. This analysis will be given in section 3. Our key results are contained in Eqs. (14), (19), and (23).

In section 4 we will present some simple illustrations of our general analysis in models taken from the literature. We will also show in these examples how the extra terms in $M_{\nu\nu}$ beyond the usual see-saw formula of Eq. (1) can resolve the puzzle of why $M_{\nu\nu}$ is not hierarchical even though m_ν probably is. In section 5, we give our conclusions and also discuss the similarities and differences with the recent work of Nir and Shadmi in [7].

2 Effective Dirac Mass Matrices and Hierarchies

Before we turn to the problem of neutrino mass let us review how the effective mass matrices of the three observed families of quarks and charged leptons arise from integrating out superheavy vectorlike fermions. What we shall call the "effective Dirac mass matrix of the neutrinos", which is important for our later analysis, arises in exactly the same way. This effective Dirac mass matrix of the neutrinos is closely related by group theory in most unified theories to the effective mass matrices of the quarks and charged leptons, and thus shares their hierarchical structure.

Consider fermions of type f and f^c, where f can be u, d, ℓ^-, or ν, and f^c can be u^c, d^c, ℓ^+, or N^c. The f are weak-$SU(2)$ doublets and the f^c are singlets. Imagine that in addition to the three families f_i and f^c_i $(i = 1, 2, 3)$ there are $f'_a + \overline{f}'_a$ $(a = 1, ..., n_f)$ and $f^{c\prime}_a + \overline{f^c}'_a$ $(a = 1, ..., n_{f^c})$ coming from $\mathbf{R}$. The f'_a and $f^{c\prime}_a$ have the same standard model quantum numbers as the f_i and f^c_i, while the $\overline{f}'_a$ and $\overline{f^c}'_a$ have opposite standard model quantum numbers.

Then we can have superheavy mass terms of the form

$$(f^T, f'^T) \begin{pmatrix} \hat{M}_f \\ M_f \end{pmatrix} \overline{f}' + (f^{cT}, f^{c\prime T}) \begin{pmatrix} \hat{M}_{f^c} \\ M_{f^c} \end{pmatrix} \overline{f^c}', \tag{2}$$

where we have suppressed indices. The matrix M_f is $n_f \times n_f$; $\hat{M}_f$ is $3 \times n_f$; M_{f^c} is $n_{f^c} \times n_{f^c}$; and $\hat{M}_{f^c}$ is $3 \times n_{f^c}$. It is important to note that these matrices can depend on the superheavy VEVs of adjoint and other non-singlet Higgs fields. There can also be, in the most general case, weak-$SU(2)$-breaking masses of the form

$$f^T m_f f^c + f^T m'_f f^{c\prime} + f'^T m''_f f^{c\prime} + f'^T m'''_f f^c. \tag{3}$$

To find the effective low-energy mass matrix, we must identify the light states, which we shall call f_ℓ and f^c_ℓ, and the superheavy states, which we shall call f_h and f^c_h. This can be done by unitary transformations $(f^T, f'^T) = (f^T_\ell, f^T_h)\mathcal{U}_f$ and $(f^{cT}, f^{c\prime T}) = (f^{cT}_\ell, f^{cT}_h)\mathcal{U}_{f^c}$, where

$$\mathcal{U}_K \begin{pmatrix} \hat{M}_K \\ M_K \end{pmatrix} = \begin{pmatrix} 0 \\ \overline{M}_K \end{pmatrix}, \; K = f, f^c. \tag{4}$$

Then

$$\mathcal{U}_K = \begin{pmatrix} U_K & -U_K Z_K \\ Z^\dagger_K U_K & V_K \end{pmatrix}, \tag{5}$$

where $Z_K = \hat{M}_K M^{-1}_K$, $U_K = (I + Z_K Z^\dagger_K)^{-1/2}$, $V_K = (I + Z^\dagger_K Z_K)^{-1/2}$, and $\overline{M}_K = V^{-1}_K M_K$. This transformation allows us to write $f = U^T_f f_\ell + ...$, $f' = -Z^T_f U^T_f f_\ell + ...$, $f^c = U^T_{f^c} f^c_\ell + ...$, and $f^{c\prime} = -Z^T_{f^c} U^T_{f^c} f^c_\ell + ...$, where the dots represent terms with heavy eigenstates. Then, inserting these results into Eq. (3), we may write the effective mass term of the light eigenstates as $f^T_\ell \overline{m}_f f^c_\ell$, where

$$\overline{m}_f = U_f \left(m_f - m'_f Z^T_{f^c} - Z_f m'''_f + Z_f m''_f Z^T_{f^c} \right) U^T_{f^c}. \tag{6}$$

The four terms in this expression have the diagrammatic interpretations shown in Fig. 1.

If the mixings are small, then $|Z_K| \ll 1$, and $U_K \cong I$. The first term in the effective mass matrix $\overline{m}_f$ is then $\cong m_f$, which just comes from the renormalizable operator corresponding to Fig. 1(a). This contribution is

usually assumed to give the largest elements in the quark and lepton mass matrices (e.g. the 33 elements). The other terms in $\overline{m}_f$ come from the higher-order effective operators corresponding to the other diagrams in Fig. 1. These are usually assumed to give the smaller elements in the mass matrices. If, for instance, the elements of $\hat{M}_K$ were of order ϵ times the elements of M_K, then the higher order terms in $\overline{m}_f$ would be suppressed by powers of ϵ relative to the lowest order terms. All models in which the "textures" of mass matrices are explained by tree-level diagrams fit the general scheme we have just outlined. We will consider some realistic examples taken from the literature in section 4.

It is usually assumed that the term m_f has the form $diag(0, 0, 1)$ and gives the masses of the third family. The matrices M_K and/or $\hat{M}_K$ can contain VEVs of adjoints or larger representations of Higgs fields, and thus introduce group generators into the smaller elements of $\overline{m}_f$ which can explain the Georgi-Jarlskog factors and other differences between the matrices of up quarks, down quarks, charged leptons, and neutrinos. It is important to emphasize that the discussion we have just given applies to the neutrinos as well as the other fermions, and that the matrix $\overline{m}_\nu$ given by Eq. (6) with $f = \nu$ and $f^c = N^c$ is just the effective Dirac mass matrix of the neutrinos. This is the matrix that naively would be expected to appear in the see-saw formula of Eq. (1). Whether it does appear in that formula, and whether there are corrections to it, can only be determined by analyzing the entire neutrino mass matrix, including the $(B - L)$-violating terms that give the Majorana mass matrix of the right-handed neutrinos N^c and $N^{c\prime}$. This will be done in the next section.

3 A General Analysis of Neutrino Masses

We will give the analysis in the language of $SO(10)$, though it is simple to generalize to larger groups simply by decomposing under the $SO(10)$ subgroup. We may take $\mathbf{F}_i = \mathbf{16}_i$. The real representations $\mathbf{R}$ of quarks and leptons can consist of some number of $\mathbf{1}, \mathbf{10}, (\mathbf{16} + \overline{\mathbf{16}}), \mathbf{45}, \mathbf{54}$, or even larger multiplets (though it is hard to find models in the literature where larger multiplets of quarks and leptons appear). Of the representations just listed, $(\mathbf{16} + \overline{\mathbf{16}})$ contains both singlet and doublet neutrinos and their conjugates; $\mathbf{10}$ contains only doublet neutrinos and their conjugates; and $\mathbf{1}$, $\mathbf{45}$, and $\mathbf{54}$

contain only singlet neutrinos.

We will proceed in stages from the simplest situation to the most complicated. In subsection 3.1, we assume that **R** contains only spinor-antispinor pairs, which leads to a relatively simple set of possibilities. (It should be noted that for such models the right-handed Majorana masses of the neutrinos must come from terms of the form $\mathbf{16}\ \mathbf{16}\langle\overline{\mathbf{126}}_H\rangle$.) The models of Ref. [8], for example, fall within this class. In subsection 3.2, we allow **R** to contain both spinor-antispinor pairs and vectors. The possibilities become slightly more complicated. The right-handed Majorana neutrino masses must still come from $\mathbf{16}\ \mathbf{16}\langle\overline{\mathbf{126}}_H\rangle$. The model of Ref. [9], for example, falls within this class. Finally, in subsection 3.3, we allow **R** to contain not only spinor-antispinor pairs and vectors but also the representations **1**, **45** and **54**. The right-handed Majorana masses of the neutrinos can then come either from either $\mathbf{16}\ \mathbf{16}\langle\overline{\mathbf{126}}_H\rangle$ or from non-renormalizable operators of the form $\mathbf{16}\ \mathbf{16}\ \langle\overline{\mathbf{16}}_H\rangle\langle\overline{\mathbf{16}}_H\rangle/M_{GUT}$ obtained by integrating out the singlet or tensor multiplets of fermions. The model of Ref. [10], for example, falls in this wider class.

3.1 $SO(10)$ models where R contains only spinor-antispinor pairs.

We first consider models where **R** consists of $(\mathbf{16}_a + \overline{\mathbf{16}}_a)$, $a = 1, ..., N$. In analyzing this case we will make one very weak assumption, namely that there are no **210** Higgs multiplets coupling spinors to antispinors of quarks and leptons. This means that the only Higgs multiplets that do couple spinors to antispinors will be in **1** or **45**, neither of which contain $SU(2)$-doublet components that can get VEVs.

In the case being considered, the neutrinos fall into just three groups: $\nu_i, N_i^c \in \mathbf{16}_i = \mathbf{F}_i$ $(i = 1, 2, 3)$; $\nu'_a, N_a^{c\prime} \in \mathbf{16}_a \in \mathbf{R}$ $(a = 1, .., N)$; and $\overline{\nu}'_a, \overline{N^c}'_a \in \overline{\mathbf{16}}_a \in \mathbf{R}$ $(a = 1, ..., N)$. Given our weak assumption, the most

general form of the mass matrix is (suppressing indices):

$$\left(\nu, \nu', \overline{\nu}', N^c, N^{c\prime}, \overline{N^c}'\right) \left(\begin{array}{cccccc} \mu & \mu' & \hat{M}_\nu & m_\nu & m'_\nu & 0 \\ \mu'^T & \mu'' & M_\nu & m'''_\nu & m''_\nu & 0 \\ \hat{M}^T_\nu & M^T_\nu & - & 0 & 0 & - \\ m^T_\nu & m'''^T_\nu & 0 & \mathcal{M} & \mathcal{M}' & \hat{M}_{N^c} \\ m'^T_\nu & m''^T_\nu & 0 & \mathcal{M}'^T & \mathcal{M}'' & M_{N^c} \\ 0 & 0 & - & \hat{M}^T_{N^c} & M^T_{N^c} & \tilde{\mathcal{M}} \end{array} \right) \left(\begin{array}{c} \nu \\ \nu' \\ \overline{\nu}' \\ N^c \\ N^{c\prime} \\ \overline{N^c}' \end{array} \right). \tag{7}$$

The dashes represent entries that can be neglected because they lead to contributions to $M_{\nu\nu}$ higher order than M_W^2/M_{GUT}. The matrices denoted with lower-case m come from $SU(2)$-breaking vacuum expectation values (VEVs) that couple **16** to **16** (m_ν, m'_ν, m''_ν, m'''_ν). The matrices denoted by capital M are GUT-scale masses that couple **16** to $\overline{\mathbf{16}}$ (M_ν, $\hat{M}_\nu$, M_{N^c}, $\hat{M}_{N^c}$). The preceding matrices are of types considered in section 2. However, the matrices denoted by calligraphic $\mathcal{M}$ were not considered in section 2. They are the GUT-scale, $(B-L)$-violating "right-handed Majorana mass matrices" and come from terms of the form **16 16** $\langle \overline{\mathbf{126}}_H \rangle$ (for $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$, $\mathcal{M}'''$) or of the form $\overline{\mathbf{16}}$ $\overline{\mathbf{16}}$ $\langle \mathbf{126}_H \rangle$ (for $\tilde{\mathcal{M}}$). The matrices denoted by μ are $O(M_W^2/M_{GUT})$ masses coming from $SU(2)$-triplet VEVs. These (if they exist) are the so-called type II see-saw contributions

If we set the $(B-L)$-violating masses ($\mathcal{M}$ etc.) to zero, we will end up with the situation already analyzed in section 2. That is, we will end up (after integrating out superheavy fields) with three light ν and three light N^c connected by a weak-scale effective Dirac mass matrix $\overline{m}_\nu$ given by Eq. (6) with $f = \nu$ and $f^c = N^c$. To see this, one simply transforms the full matrix in Eq. (7) (with $\mathcal{M}$ etc. set equal to zero) by multiplying it from the left by $\mathcal{U}$ and from the right by $\mathcal{U}^T$, where

$$\mathcal{U} = \left(\begin{array}{cccccc} U_\nu & -U_\nu Z_\nu & 0 & 0 & 0 & 0 \\ Z^\dagger_\nu U_\nu & V_\nu & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{N^c} & -U_{N^c} Z_{N^c} & 0 \\ 0 & 0 & 0 & Z^\dagger_{N^c} U_{N^c} & V_{N^c} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array} \right). \tag{8}$$

See Eqs. (4) and (5). If one makes the same transformation *without* setting the $(B-L)$-violating masses ($\mathcal{M}$ etc.) to zero, the matrix in Eq. (7) takes

the form

$$\begin{pmatrix} \overline{\mu} & - & 0 & \overline{m}_\nu & \overline{m'}_\nu & 0 \\ - & - & \overline{M}_\nu & - & - & 0 \\ 0 & \overline{M}_\nu^T & - & 0 & 0 & - \\ \overline{m}_\nu^T & - & 0 & \overline{\mathcal{M}} & \overline{\mathcal{M}'} & 0 \\ \overline{m'}_\nu^T & - & 0 & \overline{\mathcal{M}'}^T & \overline{\mathcal{M}''} & \overline{M}_{N^c} \\ 0 & 0 & - & 0 & \overline{M}_{N^c}^T & \tilde{\mathcal{M}} \end{pmatrix}. \tag{9}$$

Note that the matrices shown in the lower-right 5×5 block have GUT-scale entries, while the matrices in the first row and column have entries that are weak-scale or smaller. The dashes represent matrices that can be neglected. The matrix $\overline{m}_\nu$ is the same as that given in Eq. (6) with $f = \nu$ and $f^c = N^c$:

$$\overline{m}_\nu = U_\nu \left(m_\nu - m'_\nu Z_{N^c}^T - Z_\nu m'''_\nu + Z_\nu m''_\nu Z_{N^c}^T \right) U_{N^c}^T. \tag{10}$$

That is, it is just the "effective Dirac neutrino mass matrix." The matrix $\overline{m'}_\nu$ is given by

$$\overline{m'}_\nu = U_\nu \left(m'_\nu + m_\nu Z_{N^c}^* - Z_\nu m''_\nu - Z_\nu m''' Z_{N^c}^* \right) V_{N^c}^T. \tag{11}$$

For later reference, we give here the expressions for the remaining barred matrices: $\overline{\mathcal{M}} = U_{N^c}(\mathcal{M} - \mathcal{M}' Z_{N^c}^T - Z_{N^c} \mathcal{M}'^T + Z_{N^c} \mathcal{M}'' Z_{N^c}^T) U_{N^c}^T$; $\overline{\mathcal{M}'} = U_{N^c}(\mathcal{M}' + \mathcal{M} Z_{N^c}^* - Z_{N^c} \mathcal{M}'' - Z_{N^c} \mathcal{M}'^T Z_{N^c}^*) V_{N^c}^T$; $\overline{\mathcal{M}''} = V_{N^c}(\mathcal{M}'' + \mathcal{M}'^T Z_{N^c}^* + Z_{N^c}^\dagger \mathcal{M}' + Z_{N^c}^\dagger \mathcal{M} Z_{N^c}^*) V_{N^c}^T$.

From Eq. (9) we see that the effective mass matrix of the three light neutrinos is given by

$$M_{\nu\nu} = - \left(\overline{m}_\nu, \overline{m'}_\nu, 0 \right) \begin{pmatrix} \overline{\mathcal{M}} & \overline{\mathcal{M}'} & 0 \\ \overline{\mathcal{M}'}^T & \overline{\mathcal{M}''}^T & \overline{M}_{N^c} \\ 0 & \overline{M}_{N^c}^T & \tilde{\mathcal{M}} \end{pmatrix}^{-1} \begin{pmatrix} \overline{m}_\nu^T \\ \overline{m'}_\nu^T \\ 0 \end{pmatrix} \; + \; \overline{\mu}. \tag{12}$$

Let us first consider an easy case.

Case I: $\tilde{\mathcal{M}} = 0$. (That is, there are no GUT-scale, $(B-L)$-violating mass terms of the form $\overline{\mathbf{16}}\,\overline{\mathbf{16}} \langle \mathbf{126}_H \rangle$.) In this case, the inverse of the 3×3 block matrix in Eq. (12) is easily seen to have vanishing "12", "21", and "22" entries. So one obtains the standard see-saw formula together with possible type II corrections:

$$M_{\nu\nu} = -\overline{m}_\nu \overline{\mathcal{M}}^{-1} \overline{m}_\nu^T \; + \; \overline{\mu}. \tag{13}$$

Note the important fact that not only does $M_{\nu\nu}$ have the standard see-saw form, but the Dirac mass matrix $\overline{m}_\nu$ appearing in the formula is just the effective Dirac mass matrix that one would compute by setting the "right-handed Majorana masses" (which violate $B-L$) to zero and integrating out the extra neutrinos and antineutrinos in $\mathbf{R}$. That is, it is the same Dirac neutrino matrix that is related by the unified symmetries to the effective mass matrices of the up quarks, down quarks, and charged leptons, and is like them hierarchical. In other words, in Case I the naive procedure for computing $M_{\nu\nu}$ is correct.

Case II: $\tilde{\mathcal{M}} \neq 0$. (That is, there do exist $(B-L)$-violating terms of the form $\overline{\mathbf{16}}\ \overline{\mathbf{16}}\langle \mathbf{126}_H \rangle$.) Here the inversion of the matrix in Eq. (12) can still be done and yields the result

$$M_{\nu\nu} = -\overline{m}_\nu \overline{\mathcal{M}}^{-1} \overline{m}_\nu^T \ + \ \overline{\mu} \ - \ m_* \mathcal{M}_*^{-1} m_*^T, \tag{14}$$

where

$$m_* \equiv \overline{m'}_\nu - \overline{m}_\nu \overline{\mathcal{M}}^{-1} \overline{\mathcal{M}'}, \tag{15}$$

and

$$\mathcal{M}_* \equiv \overline{\mathcal{M}''} - \overline{\mathcal{M}'}^T \overline{\mathcal{M}}^{-1} \overline{\mathcal{M}'} - \overline{M}_{N^c} \tilde{\mathcal{M}}^{-1} \overline{M}_{N^c}^T. \tag{16}$$

The three terms in Eq. (14) have the diagrammatic interpretation given in Fig. 2. The third term of Eq. (14) looks complicated, but that is because we are considering very general possibilities. In realistic models, many of the submatrices in the mass matrix in Eq. (7) vanish, so that Eqs. (15) and (16) often reduce to quite simple expressions, as we shall see in section 4. We shall also see that the third term in Eq. (14) can explain why $M_{\nu\nu}$ is not hierarchical. Note that in the limit $\tilde{\mathcal{M}} \longrightarrow 0$, one has $\mathcal{M}_*^{-1} \longrightarrow 0$, so that one recovers Case I.

3.2 $SO(10)$ models where R contains only spinor-antispinor pairs and vectors.

We now consider models where $\mathbf{R}$ consists of $(\mathbf{16}_a + \overline{\mathbf{16}}_a)$, $a = 1, ..., N$, and $\mathbf{10}_m$, $m = 1, ..., M$. The vectors $\mathbf{10}_m$ contain additional ν' and $\overline{\nu}'$. There can be the following new kinds of terms: (a) $M_{mn}(\mathbf{10}_m \mathbf{10}_n)$. This gives new contributions to the matrix we have been calling M_ν. (b) $(\mathbf{10}_m \mathbf{16}_{i,a})\langle \mathbf{16}_H \rangle$. This can give new contributions to M_ν (or $\hat{M}_\nu$), if we take the $\overline{\nu}'$ component

of $\mathbf{10}_m$ and the ν (or ν') component of the $\mathbf{16}_i$ (or $\mathbf{16}_a$). It can also give weak-scale masses connecting the $\overline{\nu}'$ component of $\mathbf{10}_m$ to the N^c (or $N^{c\prime}$) component of the $\mathbf{16}_i$ (or $\mathbf{16}_a$). However, since these lead to negligible contributions to $M_{\nu\nu}$, they are represented by dashes below. (c) $(\mathbf{10}_m\overline{\mathbf{16}}_a)\langle\overline{\mathbf{16}}_H\rangle$. This can give new contributions to M_ν, but also a new and significant kind of contribution, namely to the submatrix that couples ν' in the vector(s) to the $\overline{N^c}'$ in the antispinor(s). This entry will be denoted below by the matrix $\tilde{m}$. Altogether, then, the matrix has almost the same form as in Eq.(7):

$$\left(\nu, \nu', \overline{\nu}', N^c, N^{c\prime}, \overline{N^c}'\right) \begin{pmatrix} \mu & \mu' & \hat{M}_\nu & m_\nu & m'_\nu & 0 \\ \mu'^T & \mu'' & M_\nu & m'''_\nu & m''_\nu & \tilde{m} \\ \hat{M}_\nu^T & M_\nu^T & - & - & - & - \\ m_\nu^T & m_\nu'''^T & - & \mathcal{M} & \mathcal{M}' & \hat{M}_{N^c} \\ m_\nu'^T & m_\nu''^T & - & \mathcal{M}'^T & \mathcal{M}'' & M_{N^c} \\ 0 & \tilde{m}^T & - & \hat{M}_{N^c} & M_{N^c}^T & \tilde{\mathcal{M}} \end{pmatrix} \begin{pmatrix} \nu \\ \nu' \\ \overline{\nu}' \\ N^c \\ N^{c\prime} \\ \overline{N^c}' \end{pmatrix}. \tag{17}$$

If the matrix we are calling $\tilde{m}$ vanishes, then the form of this matrix is not different, except for negligible entries, from the matrix we considered in the previous subsection. So the results from that analysis still apply, including Eqs. (13) and (14). However, if $\tilde{m} \neq 0$, there are new cases to consider.

Case III: $\tilde{m} \neq 0$ and $\tilde{\mathcal{M}} \neq 0$. The general expression for $M_{\nu\nu}$ is easily found to be

$$M_{\nu\nu} = -\left(\overline{m}_\nu, \overline{m'}_\nu, -U_\nu Z_\nu \tilde{m}\right) \begin{pmatrix} \overline{\mathcal{M}} & \overline{\mathcal{M}'} & 0 \\ \overline{\mathcal{M}'}^T & \overline{\mathcal{M}''}^T & \overline{M}_{N^c} \\ 0 & \overline{M}_{N^c}^T & \tilde{\mathcal{M}} \end{pmatrix}^{-1} \begin{pmatrix} \overline{m}_\nu^T \\ \overline{m'}_\nu^T \\ -\tilde{m}^T Z_\nu^T U_\nu^T \end{pmatrix} + \overline{\mu}. \tag{18}$$

This case is very complicated in general. However, there is a subcase that is relatively simple, namely if the elements of $\overline{M}_{N^c}$ are very small compared to the $(B-L)$-violating entries ($\tilde{\mathcal{M}}$, $\mathcal{M}$, etc.) then we can approximate the "33" component of the inverse matrix in Eq. (18) by $\tilde{\mathcal{M}}^{-1}$, giving

$$\begin{aligned} M_{\nu\nu} &\cong -\overline{m}_\nu \overline{\mathcal{M}}^{-1} \overline{m}_\nu^T + \overline{\mu} - m_* \mathcal{M}_*^{-1} m_*^T \\ &- U_\nu \left(\hat{M}_\nu M_\nu^{-1} \tilde{m} \tilde{\mathcal{M}}^{-1} \tilde{m}^T M_\nu^{T-1} \hat{M}_\nu^T\right) U_\nu^T, \end{aligned} \tag{19}$$

Where the first three terms are the same as in Eq. (14), and the quantities with asterisk subscripts are still defined by Eqs. (15) and (16). The last term

in Eq. (19) is the contribution from $\tilde{m}$.

3.3 $SO(10)$ models where R contains spinor-antispinor pairs, vectors, singlets and tensors

We now introduce, in addition to spinor-antispinor pairs ($\mathbf{16}+\overline{\mathbf{16}}$) and vectors $\mathbf{10}$, $SO(10)$ singlet and tensor multiplets. We shall only allow the rank-2 tensors $\mathbf{45}$ and $\mathbf{54}$. (It is unusual to find multiplets of quarks and leptons larger than these in published models.) The $\mathbf{1}$, $\mathbf{45}$, and $\mathbf{54}$ contain leptons that are singlets under the standard model group, which we shall denote by S_I. The existence of at least three such singlets allows the "right-handed Majorana mass matrix" of the neutrinos to be generated by effective operators of the form $\mathbf{16}\ \mathbf{16}\langle\overline{\mathbf{16}}_H\rangle\langle\overline{\mathbf{16}}_H\rangle/M_{GUT}$, which come from integrating the singlets out. The existence of operators of the form $\mathbf{16}\ \mathbf{16}\langle\overline{\mathbf{126}}_H\rangle$ would therefore be unnecessary.

The general case of neutrino mass where $\mathbf{R}$ contains spinor-antispinor pairs, vectors, singlets and small tensors of $SO(10)$ is very complicated. So we shall restrict our attention to a subcase defined by the following assumptions: (1) There are exactly three singlets S_I. (2) There are no $\overline{\mathbf{126}}$ or $\mathbf{126}$ of Higgs fields. (3) There are no $SU(2)$-breaking masses from terms of the form $\mathbf{10}_m\overline{\mathbf{16}}_a\langle\overline{\mathbf{16}}_H\rangle$. i.e. in the notation of the previous subsection, $\tilde{m}=0$.

With these assumptions, the following new types of terms are allowed: (a) $M\mathbf{1}\ \mathbf{1}$, $M\mathbf{45}\ \mathbf{45}$, and $M\mathbf{54}\ \mathbf{54}$. These lead to the mass matrix M_S below. (b) $\mathbf{10}\ \mathbf{1}\langle\mathbf{10}_H\rangle$, and similar terms with the singlet replaced by a tensor. These couple S to ν' and $\overline{\nu}'$, giving the entries denoted g' and $\tilde{g}$ below. (c) $\mathbf{16}\ \mathbf{1}\langle\overline{\mathbf{16}}_H\rangle$, and similar terms with the singlet replaced by a tensor. These couple S to ν, N^c, ν', and $N^{c\prime}$, giving the entries denoted g, G, g', and G' below. (d) $\overline{\mathbf{16}}\ \mathbf{1}\langle\mathbf{16}_H\rangle$, and similar terms with the singlet replaced by a tensor. These couple S to $\overline{\nu}'$ and $\overline{N^c}'$, giving the terms denoted by $\tilde{g}$ and $\tilde{G}$ below.

Given assumption (2), the kinds of masses we denoted above by $\mathcal{M}$, $\mathcal{M}'$, $\mathcal{M}''$, and $\tilde{\mathcal{M}}$ all vanish, as do those we denoted by μ, μ', etc. The general

form of the mass matrix consistent with these assumptions is thus

$$\left(\nu,\nu',\overline{\nu}',N^c,N^{c\prime},\overline{N^c}',S\right)\begin{pmatrix} 0 & 0 & \hat{M}_\nu & m_\nu & m'_\nu & 0 & g \\ 0 & 0 & M_\nu & m'''_\nu & m''_\nu & 0 & g' \\ \hat{M}_\nu^T & M_\nu^T & 0 & 0 & 0 & - & \tilde{g} \\ m_\nu^T & m'''^T_\nu & 0 & 0 & 0 & \hat{M}_{N^c} & G \\ m'^T_\nu & m''^T_\nu & 0 & 0 & 0 & M_{N^c} & G' \\ 0 & 0 & - & \hat{M}^T_{N^c} & M^T_{N^c} & 0 & \tilde{G} \\ g^T & g'^T & \tilde{g}^T & G^T & G'^T & \tilde{G}^T & M_S \end{pmatrix}\begin{pmatrix} \nu \\ \nu' \\ \overline{\nu}' \\ N^c \\ N^{c\prime} \\ \overline{N^c}' \\ S \end{pmatrix}. \tag{20}$$

Rotating the basis by the same unitary transformation in Eq. (8) (leaving the singlets S alone) one obtains the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \overline{m}_\nu & \overline{m'}_\nu & 0 & \overline{g} \\ 0 & 0 & \overline{M}_\nu & - & - & 0 & - \\ 0 & \overline{M}_\nu^T & 0 & 0 & 0 & - & - \\ \overline{m}_\nu^T & - & 0 & 0 & 0 & 0 & \overline{G} \\ \overline{m'}^T_\nu & - & 0 & 0 & 0 & M_{N^c} & \overline{G'} \\ 0 & 0 & - & 0 & \overline{M}^T_{N^c} & 0 & \tilde{G} \\ \overline{g}^T & - & - & \overline{G}^T & \overline{G'}^T & \tilde{G}^T & M_S \end{pmatrix}. \tag{21}$$

The light neutrino mass matrix is thus given by

$$M_{\nu\nu} = -(\overline{m}_\nu, \overline{m'}_\nu, 0, \overline{g})\begin{pmatrix} 0 & 0 & 0 & \overline{G} \\ 0 & 0 & \overline{M}_{N^c} & \overline{G'} \\ 0 & \overline{M}^T_{N^c} & 0 & \tilde{G} \\ \overline{G}^T & \overline{G'}^T & \tilde{G}^T & M_S \end{pmatrix}^{-1}\begin{pmatrix} \overline{m}^T_\nu \\ \overline{m'}^T_\nu \\ 0 \\ \overline{g}^T \end{pmatrix}, \tag{22}$$

which yields

$$M_{\nu\nu} = -\overline{m}_\nu \mathcal{M}^{-1}_{eff}\overline{m}^T_\nu - (\overline{m}_\nu\, A^T + A\,\overline{m}^T_\nu), \tag{23}$$

where

$$\begin{aligned} \mathcal{M}_{eff} &\equiv -\overline{G}^{T-1}(M_S - \overline{G'}^T\overline{M}^{T-1}_{N^c}\tilde{G} - \tilde{G}^T\overline{M}^{-1}_{N^c}\overline{G'})\overline{G}^{-1} \\ A &\equiv \overline{g}\,\overline{G}^{-1} - \overline{m'}_\nu\overline{M}^{T-1}_{N^c}\tilde{G}\overline{G}^{-1}. \end{aligned} \tag{24}$$

The first term in Eq. (23) is just the standard (or "type I") see-saw contribution. The remaining terms in Eq. (23) have the so-called "type III" see-saw

form. The first term in A is, in fact, exactly the type III contribution discussed in Ref. [5], while the second term in A is an additional contribution coming from the presence of the extra spinor-antispinor pairs and vectors.

By relaxing some of the assumptions we made at the beginning of this subsection, one can have more complicated situations, in which there are contributions of all types: type I, type II, type III, and the additional contributions coming from $\tilde{\mathcal{M}} \neq 0$ and $\tilde{m} \neq 0$ discussed previously. However, in most models one does not have the most general possible situation.

4 Examples from Realistic Published Models

4.1 A model where R contains only spinor-antispinor pairs

In Ref. [8] an approach to understanding quark and lepton mass hierarchies is proposed that is based on $SO(10)$ and the Froggatt-Nielson idea [11]. A family $U(1)$ is introduced that is broken by the VEVs of familons. These familons are in singlets or adjoints of the $SO(10)$. There are extra vectorlike multiplets of quarks and leptons consisting of equal numbers of spinors and antispinors. When these superheavy spinors and antispinors are integrated out, one is left with effective Yukawa operators that involve various powers of the familon field VEVs. The operators that are of higher order in these VEVs are more suppressed, giving rise to a hierarchy of masses.

In Ref. [8] a set of criteria is imposed that leads to nine realistic models of the type just described. These models are specified in that paper by the form of the effective Yukawa operators. The details of how these operators arise are not described in detail in that paper, except in the case of "model 9". (See Appendix A of Ref. [8].) We shall take as our example a simplified version of model 9, in which the operators that contribute to the masses of the lightest family (and the fields that must be integrated out to produce these operators) are neglected. This simplified version will be sufficient for our purposes.

The quark and lepton content of the model is the following, with the family charges given in parentheses: $\mathbf{16}_1(2)$, $\mathbf{16}_2(1)$, $\mathbf{16}_3(0)$, $\mathbf{16}'_1(-1)$, $\mathbf{16}'_2(-\frac{1}{2})$, $\overline{\mathbf{16}}'_1(\frac{3}{2})$, $\overline{\mathbf{16}}'_2(1)$. The weak $SU(2)_L \times U(1)_Y$ breaking is done by a vector: $\mathbf{10}_H$. The familons consist of two adjoint Higgs fields $\mathbf{45}_{(B-L)}(-1)$, $\mathbf{45}_X(-\frac{1}{2})$, and

a singlet $\mathbf{1}_H(-2)$. The adjoint $\mathbf{45}_{(B-L)}$ has VEV proportional to the generator $B-L$, while $\mathbf{45}_X$ has VEV proportional to the generator X, where $SO(10) \supset SU(5) \times U(1)_X$. We will assume that all Yukawa couplings are of order unity, that $\langle \mathbf{45}_X \rangle \sim M_{GUT}$, and that $\langle \mathbf{45}_{(B-L)} \rangle \sim \epsilon M_{GUT}$, and $\langle \mathbf{1}_H \rangle \sim \delta M_{GUT}$, where $\delta \ll \epsilon \ll 1$.

With these fields, one has the couplings: (a) $\mathbf{16}'_1 \overline{\mathbf{16}}'_1 \langle \mathbf{45}_X \rangle$, $\mathbf{16}'_2 \overline{\mathbf{16}}'_2 \langle \mathbf{45}_X \rangle$, and $\mathbf{16}'_2 \overline{\mathbf{16}}'_1 \langle \mathbf{45}_{(B-L)} \rangle$, which give the matrices M_f; (b) $\mathbf{16}_3 \overline{\mathbf{16}}'_2 \langle \mathbf{45}_{(B-L)} \rangle$ and $\mathbf{16}_2 \overline{\mathbf{16}}'_2 \langle \mathbf{1}_H \rangle$, which give the matrices $\hat{M}_f$; (c) $\mathbf{16}_3 \mathbf{16}_3 \langle \mathbf{10}_H \rangle$, which gives the matrices m_f; and (d) $\mathbf{16}_2 \mathbf{16}'_1 \langle \mathbf{10}_H \rangle$, which gives the matrices $m'_f = m'''^T_f$. (Here the subscript f stands for any of the types of fermion or antifermion.) We may then write the various mass matrices in the following form:

$$
\begin{aligned}
m_f &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_{U,D}, \\
m'_f &= m'''^T_f = \begin{pmatrix} 0 & 0 \\ x & 0 \\ 0 & 0 \end{pmatrix} m_{U,D}, \quad m''_f = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
M_f &= \begin{pmatrix} X_f & 0 \\ (B-L)_f \epsilon y & X_f \end{pmatrix} M, \quad \hat{M}_f = \begin{pmatrix} 0 & 0 \\ 0 & \delta \\ 0 & (B-L)_f \epsilon \end{pmatrix} M.
\end{aligned}
\tag{25}
$$

The weak-scale masses denoted m_U and m_D are proportional, respectively, to the VEVs v_u and v_d. The mass M is of order the unification scale. $x, y \sim 1$. The foregoing allows us to write $Z_f \equiv \hat{M}_f M_f^{-1}$ as

$$
Z_f = \begin{pmatrix} 0 & 0 \\ -\frac{(B-L)_f}{X_f^2} \delta \epsilon y & \frac{1}{X_f} \delta \\ -\frac{(B-L)_f^2}{X_f^2} \epsilon^2 y & \frac{(B-L)_f}{X_f} \epsilon \end{pmatrix}. \tag{26}
$$

We shall neglect all terms that are subleading in δ and ϵ. That means that we may treat the matrices U_f and V_f as being equal to the identity. The effective mass matrices $\overline{m}_f$ of the light states are then given by $\overline{m}_f \cong$

$m_f - m'_f Z^T_{f^c} - Z_f m'^T_f$. (See Eqs. (6) and (25).) This gives

$$\overline{m}_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\frac{(B-L)_f}{X_f^2} + \frac{(B-L)_{f^c}}{X_{f^c}^2}\right)\delta\epsilon xy & \frac{(B-L)^2_{f^c}}{X^2_{f^c}}\epsilon^2 xy \\ 0 & \frac{(B-L)^2_f}{X^2_f}\epsilon^2 xy & 1 \end{pmatrix} m_{U,D}. \tag{27}$$

In particular, for the up quark, down quark, neutrino Dirac, and charged lepton mass matrices, one has

$$\begin{aligned} \overline{m}_u &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9}\epsilon^2 xy \\ 0 & \frac{1}{9}\epsilon^2 xy & 1 \end{pmatrix} m_U, & \overline{m}_d &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{8}{27}\delta\epsilon xy & \frac{1}{81}\epsilon^2 xy \\ 0 & \frac{1}{9}\epsilon^2 xy & 1 \end{pmatrix} m_D \\ \overline{m}_\nu &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{16}{225}\delta\epsilon xy & \frac{1}{25}\epsilon^2 xy \\ 0 & \frac{1}{9}\epsilon^2 xy & 1 \end{pmatrix} m_U, & \overline{m}_\ell &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{8}{9}\delta\epsilon xy & \epsilon^2 xy \\ 0 & \frac{1}{9}\epsilon^2 xy & 1 \end{pmatrix} m_D \end{aligned} \tag{28}$$

The diagrams giving the 33, 23, 32, and 22 elements of $\overline{m}_\nu$ are shown in Figure 3. We see illustrated in Eq. (28) the general facts, noted earlier, that in unified models the effective Dirac mass matrix of the neutrinos $\overline{m}_\nu$, given by Eq. (6), is related by the grand unified symmetries to the effective mass matrices of the quarks and charged leptons and is hierarchical in form, like them.

We are now in a position to determine the form of the mass matrix $M_{\nu\nu}$ of the light neutrinos. The model under consideration falls in the class considered in section 3.1. Thus $M_{\nu\nu}$ is given by Eq. (14). Since we are not assuming that $SU(2)$-triplet Higgs fields acquire VEVs, we may set $\overline{\mu} = 0$. If the matrix $\tilde{\mathcal{M}}$ vanishes, then as shown in section 3.1 the term $m_* \mathcal{M}_*^{-1} m_*^T$ vanishes, and the expression for $M_{\nu\nu}$ reduces to the usual see-saw form. It should be emphasized that this is not obvious by inspecting diagrams. For instance, if the matrix $\mathcal{M}'' \neq 0$, there are diagrams (shown in Fig. 4(a)) that do not involve the elements of $\tilde{\mathcal{M}}$ but that nevertheless appear naively to give extra contributions to $M_{\nu\nu}$. However, our analysis shows (surprisingly) that such diagrams produce nothing beyond the usual see-saw terms.

Let us now look at some concrete cases. Suppose for simplicity that $\mathcal{M}'$ and $\mathcal{M}''$ vanish. Then, from the general expressions for the barred matrices (see the equations in text after Eq. (11)), and neglecting terms higher order

in the small quantities ϵ and δ, one has that $\overline{\mathcal{M}} = \mathcal{M}$, $\overline{\mathcal{M}'} = \mathcal{M}Z^*_{N^c}$, and $\overline{\mathcal{M}''} = 0$. Now consider some simple forms for $\mathcal{M}$. The easiest form to achieve in a model is skew-diagonal, as that can arise from the VEV of a single $\overline{\mathbf{126}}_H$ with family charge -2. (The relevant terms are $(\mathbf{16}_1\mathbf{16}_3 + \mathbf{16}_2\mathbf{16}_2)\overline{\mathbf{126}}_H$.) If we assume that $\mathcal{M} = \mathrm{skewdiag}(A, 1, A)M_R$, with $A \sim 1$, Eq. (13) would give

$$M_{\nu\nu} = \frac{m_U^2}{M_R}\left(\frac{\epsilon xy}{9}\right)^2 \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & (\frac{16}{25})^2\delta^2 & -\frac{16}{25}\delta\epsilon \\ 0 & -\frac{16}{25}\delta\epsilon & \epsilon^2 \end{array} \right). \tag{29}$$

Another simple possibility, in the sense of having few parameters, is a diagonal form. If we assume that $\mathcal{M} = \mathrm{diag}(B, A, 1)M_R$, with $A, B \sim 1$, Eq. (13) would give

$$M_{\nu\nu} = \frac{m_U^2}{M_R} \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & O(\delta^2\epsilon^2, \epsilon^4) & \frac{1}{25}\epsilon^2 xy \\ 0 & \frac{1}{25}\epsilon^2 xy & 1 \end{array} \right). \tag{30}$$

Note, in both cases, that the hierarchical structure of $\overline{m}_\nu$ leads to a hierarchical structure for $M_{\nu\nu}$ This can be avoided if there is a strong hierarchy in the parameters of $\mathcal{M}$ that cancels out the hierarchy in the Dirac mass matrix. (In that case some terms we dropped because they were higher order in small parameters might have to be retained as they could be multiplied by large quantities coming from $\mathcal{M}^{-1}$.) However, this requires some conspiracy between the Dirac and Majorana mass matrices in the see-saw formula.

More interesting is the possibility that the extra term in Eq. (14) could lead to a non-hierarchical pattern for $M_{\nu\nu}$. Let us then compute m_* and $\mathcal{M}_*$. We have already found that $\overline{m}_\nu = m_\nu + O(\delta, \epsilon)$. From Eq. (11), $\overline{m'}_\nu = m'_\nu + m_\nu Z^*_{N^c}$, neglecting terms higher order in small quantities. Therefore, from Eq. (15), and using the fact that $\overline{\mathcal{M}}^{-1}\overline{\mathcal{M}'} = Z^*_{N^c}$, one has $m_* = (m'_\nu + m_\nu Z^*_{N^c}) - (m_\nu + O(\epsilon, \delta))Z^*_{N^c}$. So $m_* = m'_\nu$ if we neglect terms higher order in small quantities.

From Eq. (16), $\mathcal{M}_* = -M_{N^c}\tilde{\mathcal{M}}^{-1}M^T_{N^c} + O(\epsilon^2, \delta^2)$. For $\tilde{\mathcal{M}}$ we choose the simple form $\tilde{\mathcal{M}} = \mathrm{diag}(\tilde{\mathcal{M}}_{11}, \tilde{\mathcal{M}}_{22})$. Then, simply multiplying gives

$$\delta M_{\nu\nu} = m_*\mathcal{M}_*^{-1}m_*^T = \frac{m_U^2}{25M^2}(\tilde{\mathcal{M}}_{11} + \frac{y^2}{25}\tilde{\mathcal{M}}_{22}) \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{31}$$

Note that this contribution to $M_{\nu\nu}$ is in no way hierarchically suppressed. The small familon VEVs ($\langle \mathbf{45}_{(B-L)} \rangle \sim \epsilon M_{GUT}$, $\langle \mathbf{1}_H \rangle \sim \delta M_{GUT}$) that suppress

some terms in the neutrino Dirac mass matrix (cf. Eq. (28) and Fig. 3) do not enter this expression, as can be understood from its diagrammatic form, shown in Fig. 4(b).

We have been ignoring the masses of the first family. These arise from an effective operator that is sixth order in a familon field that is an adjoint of $SO(10)$ and has family charge $-\frac{1}{2}$. In Appendix A of Ref. [8], this operator is shown to arise from integrating out additional spinor-antispinor pairs (in fact, five of them). These additional multiplets can can lead, in the same way that we have been discussing, to large entries for the first family in $m_* \mathcal{M}_*^{-1} m_*^T$.

4.2 A model where R contains spinor-antispinor pairs, vectors, and singlets

In the papers of Ref. [10] a very predictive $SO(10)$ models of quark and lepton masses is described. The fields that are integrated out to produce the higher-dimension effective Yukawa operators that give mass to the light families include spinors, vectors, and singlets. This model therefore falls into the category studied in section 3.3. As in the previous example studied, we shall for the sake of simplicity ignore the masses of the first family and the fields that are integrated out to generate them.

In the model of Ref. [10], the quark and lepton content is the following, where we indicate in parentheses the names of the neutrinos and antineutrinos contained in each $SO(10)$ multiplet: $\mathbf{16}_i(\nu_i, N_i^c)$, $i = 1,2,3$; $\mathbf{16}(\nu_1', N_1^{c\prime})$; $\overline{\mathbf{16}}(\overline{\nu}_1', \overline{N^c}_1')$; $\mathbf{10}(\nu_2', \overline{\nu}_2')$; $\mathbf{1}_i(S_i)$, $i = 1,2,3$. (We have slightly simplified the model: in the original there were two vectors of quarks and leptons. We have chosen to identify them. This does not change the model significantly.) The $SU(2)_L \times U(1)_Y$ breaking is done by a vector $\mathbf{10}_H$ and a spinor $\mathbf{16}_H'$. The GUT-scale breaking is done by an adjoint $\mathbf{45}_H$ whose VEV points in the $B-L$ direction, and by a spinor $\mathbf{16}_H$ whose VEV is in the standard-model-singlet direction.

There are the following terms: (a) $M_{16}\mathbf{16}\ \overline{\mathbf{16}} + M_{10}\mathbf{10}\ \mathbf{10}$, which give the matrices M_ν and M_{N^c}; (b) $\mathbf{16}_3\overline{\mathbf{16}}\langle\mathbf{45}_H\rangle + \mathbf{16}_2\mathbf{10}\langle\mathbf{16}_H\rangle$, which give the matrices $\hat{M}_\nu$ and $\hat{M}_{N^c}$; (c) $\mathbf{16}_3\mathbf{16}_3\langle\mathbf{10}_H\rangle$, which gives the matrix m_ν; (d) $\mathbf{16}_2\mathbf{16}\langle\mathbf{10}_H\rangle$, which gives the matrices m_ν' and m_ν'''; (e) $(M_1)_{ij}\mathbf{1}_i\ \mathbf{1}_j$, which gives the matrix M_S; (f) $\mathbf{16}_i\mathbf{1}_j\langle\overline{\mathbf{16}}_H\rangle$, which gives the matrix G; and (op-

tionally) $\overline{\mathbf{16}}\,\mathbf{1}_i\langle\mathbf{16}_H\rangle$, which gives the matrix (in this case a 3×1 matrix) $\tilde{G}$. (See Eq. (20) for the definitions.)

It is simple to see that these matrices have the following forms:

$$\begin{aligned}
M_\nu &= \begin{pmatrix} M_{16} & 0 \\ 0 & M_{10} \end{pmatrix}, & M_{N^c} &= (M_{16}), \\
\hat{M}_\nu &= \begin{pmatrix} 0 & 0 \\ 0 & \gamma \\ \beta & 0 \end{pmatrix} M_{16}, & \hat{M}_{N^c} &= \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix} M_{16}, \\
m_\nu &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_U, & m''_\nu &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
m'_\nu &= \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix} m_U, & m'''_\nu &= \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix} m_U.
\end{aligned} \tag{32}$$

Then the $Z_f \equiv \hat{M}_f M_f^{-1}$ are given by

$$Z_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \gamma\frac{M_{16}}{M_{10}} \\ \beta & 0 \end{pmatrix}, \quad Z_{N^c} = \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix}. \tag{33}$$

We take the parameters β and γ to be small, and neglect terms quadratic in them, so that the matrices U_f and V_f may be taken to be unity. From Eq. (6) it is simple to compute that $\overline{m}_\nu \cong m_\nu - m'_\nu Z^T_{N^c} - Z_\nu m'''_\nu$, or (defining $\epsilon \equiv \alpha\beta$):

$$\overline{m}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon \\ 0 & -\epsilon & 1 \end{pmatrix} m_U. \tag{34}$$

There are similar matrices for the up quarks, down quarks, and charged leptons, but we will not bother to present them.

The assumptions that led to Eq. (23) hold. Thus $M_{\nu\nu}$ has the usual see-saw term, which will obviously be hierarchical, and possibly a second term of the "type III" form. Applying Eq. (24) to the present model, we have

$A = -\overline{m'}_\nu M_{16}^{-1}\tilde{G}G^{-1}$. From Eq. (11), one has $\overline{m'}_\nu \cong m'_\nu + m_\nu Z_{N^c}$. Defining $\tilde{G}G^{-1} \equiv (x_1, x_2, x_3)$, we then have altogether

$$A \cong \begin{pmatrix} 0 & 0 & 0 \\ -\alpha x_1 & -\alpha x_2 & -\alpha x_3 \\ \beta x_1 & \beta x_2 & \beta x_3 \end{pmatrix} \frac{m_U}{M_{16}}, \tag{35}$$

and thus, by Eq. (23),

$$\delta M_{\nu\nu} \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\epsilon\alpha x_3 & -\alpha x_3 \\ 0 & -\alpha x_3 & 2\epsilon\alpha^{-1} x_3 \end{pmatrix} \frac{m_U^2}{M_{16}}. \tag{36}$$

We see, again, that non-hierarchical contributions result. In particular, in the usual see-saw term, the 23 and 32 elements are of order ϵ, whereas in this extra contribution these elements are not suppressed (α has not been assumed to be small).

5 Conclusions

We have given a fairly general analysis of neutrino mass in the context of $SO(10)$. Larger groups can be analyzed by decomposing to $SO(10)$. We have seen that in grand unified models there can be additional contributions to the mass matrix of the light neutrinos besides the usual "type I" see-saw term (or the "type II" see-saw term arising from the VEVs of triplet Higgs fields [4] and "type III" see-saw term pointed out recently [5]).

The additional contributions that we have discussed arise from integrating out the extra multiplets of fermions which typically exist at the unification scale in realistic models. We have found that in certain special cases the light neutrino mass matrix does reduce to just the type I (and possibly also type II) term. These special cases include the following. (a) The extra quark and lepton multiplets at the GUT scale consist only of spinors and antispinors of $SO(10)$ and there exist no superheavy mass terms of the form $\overline{\mathbf{16}}\,\overline{\mathbf{16}}\langle\mathbf{126}_H\rangle$. (b) The extra quark and lepton multiplets consist only of spinors, antispinors, and vectors of $SO(10)$, and there are neither superheavy mass terms of the form $\overline{\mathbf{16}}\,\overline{\mathbf{16}}\langle\mathbf{126}_H\rangle$, nor weak-scale masses of the form $\mathbf{10}\,\overline{\mathbf{16}}\langle\overline{\mathbf{16}}_H\rangle$. Except in special cases, however, there are non-type I extra terms. The forms of these extra terms are given for various cases in Eqs. (14)-(16) and Eq. (19).

Typically, the Dirac mass matrix of the neutrinos that appears in the type I see-saw term is just the effective Dirac mass neutrino matrix $\overline{m}_\nu$ that is closely related by unified symmetries to the effective mass matrices of the three observed families of up quarks, down quarks, and charged leptons. Like those matrices, $\overline{m}_\nu$ is expected to be hierarchical in structure. Unless there is a "conspiracy" between the Dirac and Majorana mass matrices appearing in the type I see-saw formula, one would therefore expect the light neutrino mass matrix $M_{\nu\nu}$ to be strongly hierarchical as well. However, the neutrino oscillation data implies that the neutrino masses are not strongly hierarchical, and the solar and atmospheric angles are not small. This is somewhat of a puzzle.

The additional non-type I contributions that we have studied in this paper might explain why $M_{\nu\nu}$ is not observed to be strongly hierarchical. In fact, we have shown by examining two realistic $SO(10)$ unified schemes that exist in the literature that the non-type I contributions can indeed be large and lead to non-hierarchical patterns for $M_{\nu\nu}$. As these examples show, especially the example discussed in section 4.1, the family symmetries that are responsible in some models for the hierarchical "textures" of the quark and charged lepton mass matrices, and indeed of the neutrino Dirac mass matrix, do not in general lead to a hierarchical form for the non-type I contributions to the neutrino mass matrix.

A significant point of our analysis is that the neutrino masses "see" deeper into the structure of the theory at the unification scale than do the masses of the quarks and charged leptons. In fact, the neutrino masses can see beyond the structure that leads the other masses to have a family hierarchy and thus itself be non-hierarchical. This last point has also been made in the recent paper of Nir and Shadmi [7], with which we became familiar after completing the analysis in this paper.

The paper of Nir and Shadmi has certain obvious points of contact with the present work, although the starting points are very different. The starting point of Nir and Shadmi is family symmetry. They consider models of the Froggatt-Nielson type, in which the family hierarchies of the quarks and charged leptons are controlled by a small parameter λ_H that is the ratio of the family-symmetry-breaking scale to the masses of the vectorlike fermions that appear in Froggatt-Nielson models. They note that neutrino masses, on the other hand, being Majorana rather than Dirac, involve an additional dimensionless parameter, namely the ratio of the $(B-L)$-breaking scale

to the masses of the vectorlike fermions. Consequently, the neutrino mass matrix can have a very different kind of hierarchy than the other types of fermion have.

Our starting point has been grand unified symmetry. The question we have tried to answer is under what general conditions the usual (type I) see-saw formula is justified in grand unified theories, and when on the contrary this formula receives corrections. We have not paid attention in the general analysis of section 3 to the issue of family symmetry or the scale at which it is broken if it exists.

The approach of the present paper and of Nir and Shadmi are thus, in a sense, orthogonal. Froggatt-Nielson models need not be (and usually are not) grand unified, and grand unified models need not be (and usually are not) of the Froggatt-Nielson type. Nevertheless, a model can be both, as is the example we have analyzed in section 4.1.

The key points that are common to our paper and that of Nir and Shadmi is that vectorlike fermions typically exist in models that seek to explain family hierarchies, and that the existence of these fermions can lead to a non-hierarchical structure for the effective mass matrix of the three light neutrinos.

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Figure Captions

Fig 1. Diagrams corresponding to the four terms of Eq. (6). $\overline{m}_f$ is the effective mass matrix of the light f, f^c obtained by integrating out the superheavy multiplets $f + \overline{f}$, and $f^c + \overline{f^c}$.

Fig 2. Contributions to $M_{\nu\nu}$. (a) Usual "type I" see-saw term. (b) The "type II" see-saw term, which comes from triplets Higgs fields. (c) The extra term that can exist if there are $\overline{\mathbf{16}}\, \overline{\mathbf{16}}\, \mathbf{126}_H$ terms. See Eqs. (14)-(16).

Fig. 3. Diagrams that give the 33, 23, 32, and 22 elements of $\overline{m}_\nu$ in the model discussed in section 4.1.

Fig 4. (a) An apparent non-type-I-see-saw contribution to $M_{\nu\nu}$ in model of section 4.1, which actually vanishes if $\tilde{\mathcal{M}} = 0$.(b) A genuine non-type-I-see-saw contribution in that model.

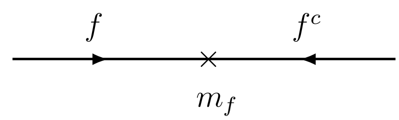


Fig. 1(a)

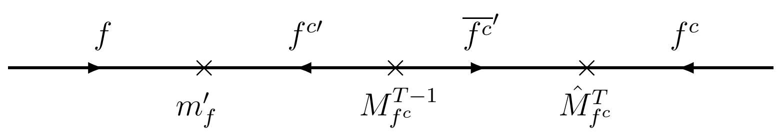


Fig. 1(b)

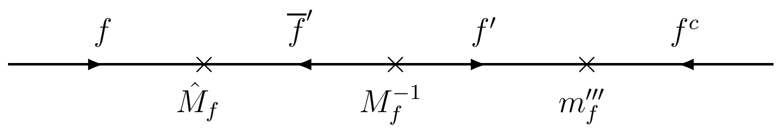

f
$\overline{f}'$
f'
f^c
$\hat{M}_f$
M_f^{-1}
m_f'''

Fig. 1(c)

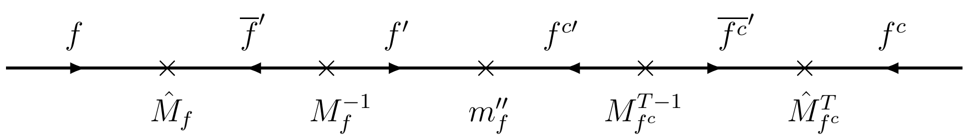

f
$\overline{f}'$
f'
$f^{c\prime}$
$\overline{f^c}'$
f^c
$\hat{M}_f$
M_f^{-1}
m_f''
$M_{f^c}^{T-1}$
$\hat{M}_{f^c}^T$

Fig. 1(d)

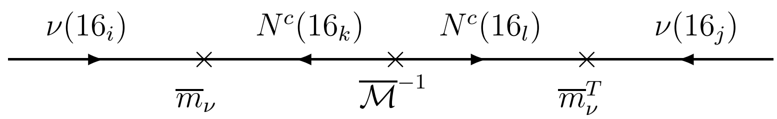

$\nu(16_i)$
$N^c(16_k)$
$N^c(16_l)$
$\nu(16_j)$
$\overline{m}_\nu$
$\overline{\mathcal{M}}^{-1}$
$\overline{m}_\nu^T$

Fig. 2(a)

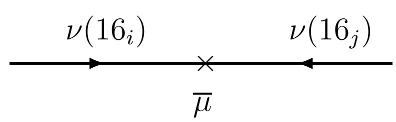

$\nu(16_i)$
$\nu(16_j)$
$\overline{\mu}$

Fig. 2(b)

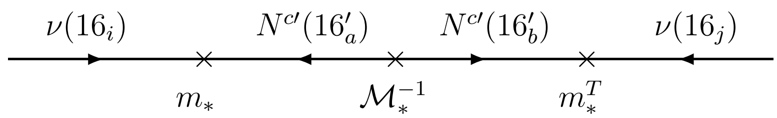


Fig. 2(c)

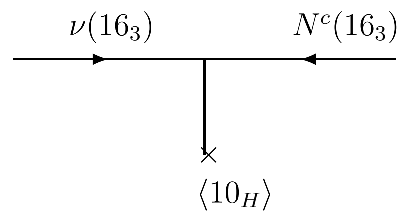

$\nu(16_3)$
$N^c(16_3)$
$\langle 10_H \rangle$

Fig. 3(a)

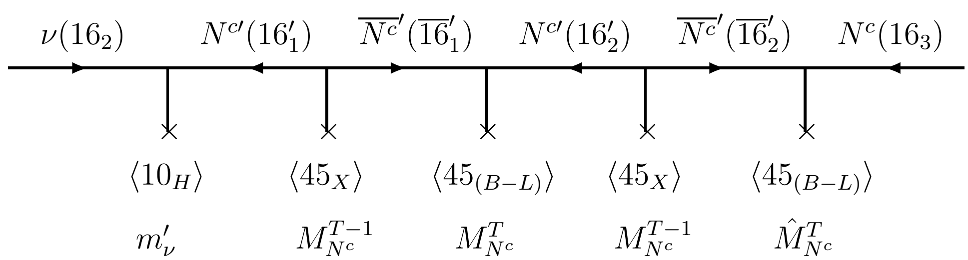

$\nu(16_2)$
$N^{c\prime}(16'_1)$
$\overline{N^c}'(\overline{16}'_1)$
$N^{c\prime}(16'_2)$
$\overline{N^c}'(\overline{16}'_2)$
$N^c(16_3)$
$\langle 10_H \rangle$
$\langle 45_X \rangle$
$\langle 45_{(B-L)} \rangle$
$\langle 45_X \rangle$
$\langle 45_{(B-L)} \rangle$
m'_ν
$M^{T-1}_{N^c}$
$M^T_{N^c}$
$M^{T-1}_{N^c}$
$\hat{M}^T_{N^c}$

Fig. 3(b)

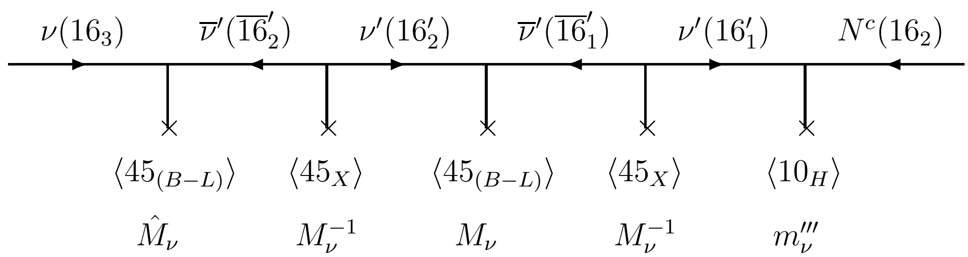

$\nu(16_3)$
$\overline{\nu}'(\overline{16}'_2)$
$\nu'(16'_2)$
$\overline{\nu}'(\overline{16}'_1)$
$\nu'(16'_1)$
$N^c(16_2)$
$\langle 45_{(B-L)} \rangle$
$\langle 45_X \rangle$
$\langle 45_{(B-L)} \rangle$
$\langle 45_X \rangle$
$\langle 10_H \rangle$
$\hat{M}_\nu$
M_ν^{-1}
M_ν
M_ν^{-1}
m'''_ν

Fig. 3(c)

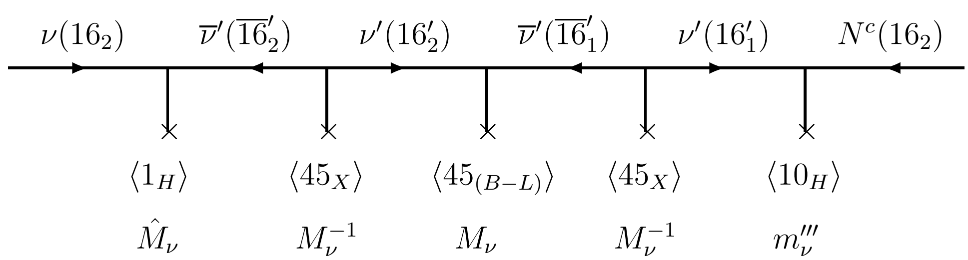
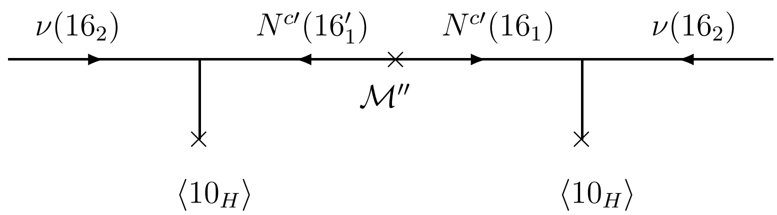
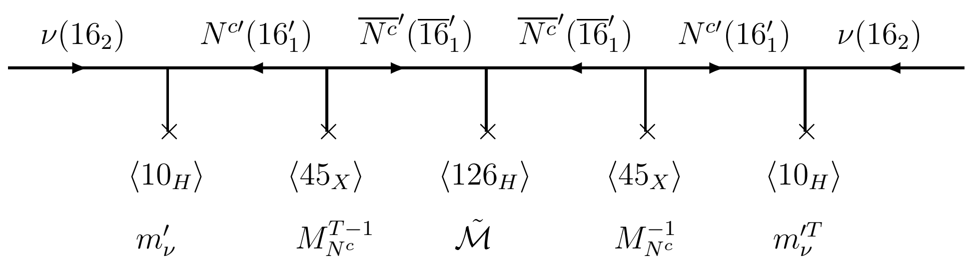

$\nu(16_2)$
$\overline{\nu}'(\overline{16}'_2)$
$\nu'(16'_2)$
$\overline{\nu}'(\overline{16}'_1)$
$\nu'(16'_1)$
$N^c(16_2)$
$\langle 1_H \rangle$
$\langle 45_X \rangle$
$\langle 45_{(B-L)} \rangle$
$\langle 45_X \rangle$
$\langle 10_H \rangle$
$\hat{M}_\nu$
M_ν^{-1}
M_ν
M_ν^{-1}
m'''_ν

Fig. 3(d)



$\nu(16_2)$
$N^{c\prime}(16_1')$
$N^{c\prime}(16_1)$
$\nu(16_2)$
$\mathcal{M}''$
$\langle 10_H \rangle$
$\langle 10_H \rangle$

Fig. 4(a)



$\nu(16_2)$
$N^{c\prime}(16_1')$
$\overline{N^c}'(\overline{16}_1')$
$\overline{N^c}'(\overline{16}_1')$
$N^{c\prime}(16_1')$
$\nu(16_2)$
$\langle 10_H \rangle$
$\langle 45_X \rangle$
$\langle 126_H \rangle$
$\langle 45_X \rangle$
$\langle 10_H \rangle$
m_ν'
$M_{N^c}^{T-1}$
$\tilde{\mathcal{M}}$
$M_{N^c}^{-1}$
$m_\nu'^T$

Fig. 4(b)